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1. If $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ and $g : (Y, \mathcal{S}) \to (Z, \mathcal{V})$ are continuous, show that the composition $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{V})$ is also continuous.

   Solution. From the definition of continuity of f, $\forall$ open sets $V \subset Y$ containing a point $f(x \in X)$, $\exists$ an open set $U \subset X$ s.t. $x \in U \implies f(x) \in V$. Similarly, for g, $\forall$ open sets $V' \subset Z$ containing a point $g(y \in Y)$, $\exists$ an open set $U' \subset Y$ s.t. $y \in U' \implies g(y) \in V'$.

   To combine the two definitions to get the desired result, take the set V as defined above to be the inverse image $g^{-1}(V')$. So now, for $g \circ f$, for an open set $V' \subset Z$ containing a point $g \circ f(x \in X)$, $\exists$ an open set $U \subset X$ s.t. $x \in U \implies g \circ f(x) \in V'$. This is the definition of continuity of a function on topologies.

   $\therefore g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{V})$ is continuous.

2. **Local base for a topological space**

   - $\mathcal{N}(x)$ is a local base for a topological space $(X, \mathcal{T})$. If $A \in \mathcal{N}(x)$, show that there exists a $U \in \mathcal{T}$ such that $x \in U \subseteq A$.

     Solution. Given $x \in X$ and $A \in \mathcal{N}(x)$, let

     $$U = \int(A) = \{y \in A | \subset V_y \in \mathcal{N}(y) \text{ s.t. } y \in V_y \subseteq A\}.$$

     Clearly, $U \subseteq A$ and $U \neq \emptyset$ since $x \in A \in \mathcal{N}(x)$ and $x \in A \subseteq A$.

     While it is clear that $y \in U \implies \exists V_y \in \mathcal{N}(y)$ s.t. $y \in V_y \subseteq A$, it is not immediately obvious that $y \in U \implies \exists V_y \in \mathcal{N}(y)$ s.t. $y \in V_y \subseteq U$, which is needed to show that $U \in \mathcal{T}$.

     To prove this, we will appeal to condition 3 in the definition of a local base By construction, $y \in U \implies \exists V_y \in \mathcal{N}(y)$ s.t. $y \in W_y \subseteq V_y$ and $z \in W_y \implies \exists P_z \in \mathcal{N}(z)$ s.t. $z \in P_z \subseteq V_y \subseteq A$. Consequently, $y \in U \implies \exists W_y \in \mathcal{N}(y)$ s.t. $z \in W_y \implies \exists P_z \in \mathcal{N}(z)$ s.t. $z \in P_z \subseteq A$, implying $W_y \subseteq U$. This proves $y \in U \implies \exists W_y \mathcal{N}(y)$ s.t. $y \in V_y \subseteq U$, so that $U \in \mathcal{T}$.

     I will close the discussion with a couple of comments. We've shown that, if $A \in \mathcal{N}(x)$, there exists $U \in \mathcal{T}$ s.t. $x \in U \subseteq A$. Conversely, since U is open and $x \in U$, there must exist a $B \in \mathcal{N}(x)$ s.t. $x \in B \subseteq U$. Given a topology $\mathcal{T}$, if we define $\mathcal{M}(x) = \{U \in \mathcal{T} | x \in U\}$, we've shown that $\mathcal{M}(x)$ is a local base for the topology $\mathcal{T}$.

     Putting all of this together, we see that there is never any loss of generality in assuming that the elements of a local base are all open in the topology generated by the local base.

- $\mathcal{N}_1(x)$ and $\mathcal{N}_2(x)$ are local bases for a space X. Show that the topology $\mathcal{T}_1$ generated by $\mathcal{N}_1(x)$ is finer that the topology $\mathcal{T}_2$ generated by $\mathcal{N}_2(x)$ if and only if for all $B \in \mathcal{N}_2(x)$, there is a set $A \in \mathcal{N}_1(x)$ such that $x \in A \subseteq B$.

  Solution. Suppose $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, or equivalently $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Set $B \in \mathbb{N}_2(x)$ for some $x \in X$. From the previous problem, there is a $U \in \mathcal{T}_2$ s.t. $x \in U \subseteq B$. This implies $U \in \mathcal{T}_1$, so there is (in particular) $A \in \mathcal{N}_1(x)$ s.t. $A \subseteq U \subseteq B$. The choice of B is arbitrary, so for all $B \in \mathcal{N}_2(x)$, there is a set $A \in \mathcal{N}_1(x)$ s.t. $x \in A \subseteq B$.

  Conversely, suppose for all $B \in \mathcal{N}_2(x)$, there is a set $A \in \mathcal{N}_1(x)$ s.t. $x \in A \subseteq B$. Let $U \in \mathcal{T}_2$. Then for all $y \in U$ there is a $B \in \mathcal{N}_2(y)$ s.t. $B \subseteq U$. This implies for all $y \in U$ there is $A \in \mathcal{N}_1(x)$ such that $A \subseteq B \subseteq U$, so $U \in \mathcal{T}_1$. Then $\mathcal{T}_2 \subseteq \mathcal{T}_1$, or equivalently $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

  $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ iff for all $B \in \mathcal{N}_2(x)$, there is a set $A \in \mathcal{N}_1(x)$ such that $x \in A \subseteq B$.

- Let $X = l^\infty(\mathbb{R}, \mathbb{N})$. For each $x \in X, m \in \mathbb{N}$, $\epsilon > 0$, let $U(x; m, \epsilon) = \{y \in X \big| \max_{1 \le i \le m} i^2 |x_i - y_i| < \epsilon\}$. If $\mathcal{N}(x) = \{U(x; m, \epsilon), m \in \mathbb{N}, \epsilon > 0\}$. Show that (i) $\mathcal{N}(x)$ is a local base, and (ii) the l^∞ metric topology is strictly finer than the topology $\mathcal{T}$ generated by this local base.

  Solution. Suppose $V \in \mathcal{N}(x)$. It follows that $x \in V$ since $\max_{1 \le i \le m} i^2 |x_i - x_i| = 0 < \epsilon$ regardless of $m \in \mathbb{N}$ and $\epsilon > 0$.

  Suppose $V_1, V_2 \in \mathcal{N}(x)$. Then $V_1 = U(x; m_1, \epsilon_1)$ and $V_2 = U(x; m_2, \epsilon_2)$ for some $m_1, m_2 \in \mathbb{N}$ and $\epsilon_1, \epsilon_2 > 0$. Let $V_3 = U(x; \max(m_1, m_2), \min(\epsilon_1, \epsilon_2) \in \mathcal{N}(x)$. If $y \in V_3$ then $y \in V_1 \cap V_2$, so $V_3 \subseteq V_1 \cap V_2$.

  Suppose $V \in \mathcal{N}(x)$. Then $V = U(x; m, \epsilon)$ for some $m \in \mathbb{N}$ and $\epsilon > 0$. Let $y \in V$ and $\delta = \max_{1 \le i \le m} i^2 |x_i - y_i| < \epsilon$. Consider $W = U(y; m(\epsilon - \delta)/2) \in \mathcal{N}(y)$. If $z \in W$ then $\max_{1 \le i \le m} i^2 |x_i - z_i| \le \max_{1 \le i \le m} i^2 (|x_i - y_i| + |y_i - z_i|) \le \max_{1 \le i \le m} i^2 |x_i - y_i| + \max_{1 \le i \le m} i^2 |y_i - z_i| < \delta + (\epsilon - \delta)/2 < \epsilon$. So $z \in V$ which implies $W \subseteq V$.

  $\therefore \mathcal{N}(x)$ is a local base.

  Suppose $V \in \mathcal{T}$. Then for all $x \in V$ there are $m \in \mathbb{N}$ and $\epsilon > 0$ such that $U(x; m, \epsilon) \subseteq V$. Let $\delta = \epsilon / m^2$ and consider the open ball in the l^∞ metric topology on X, $B_\delta(x)$. If $y \in B_\delta(x)$ then $\sup_n |x_n - y_n| < \delta$ and so $\max_{1 \le i \le m} i^2 |x_i - y_i| < \epsilon$. Thus $B_\delta(x) \subseteq U(x; m, \epsilon) \subseteq V$, which implies V is open in the l^∞ metric topology on X.

  Consider the open unit ball centered at the zero sequence in the l^∞ metric topology on X, $B_1(0)$. For any $\epsilon > 0$ and $m \in \mathbb{N}$, construct the sequence $y = \{0, 0, \ldots, 0, 2, \ldots\}$ with m zeroes. Then $y \in U(0; m, \epsilon)$ but $y \notin B_1(0)$. So, $B_1(0)$ cannot be open in $\mathcal{T}$.

  $\therefore$ the l^∞ metric topology on X is strictly finer than the topology $\mathcal{T}$ generated by $\mathcal{N}(x)$.

- The topology $\mathcal{T}$ is as defined in the previous item. Show that $x^{(n)} \in X$ converges to z in $(X, \mathcal{T})$ if and only if $x_i^{(n)} \to z_i$ for all i.

Solution. Suppose $x^{(n)} \in X$ converges to z in $(X, \mathcal{T})$. Then for all $m \in \mathbb{N}$ and $\epsilon > 0$ there is a N s.t. $x^{(n)} \in U(z; m, \epsilon)$ for all $n > N$. In particular, for some $\delta > 0$ and $i \in \mathbb{N}$, there must be N s.t. $x^{(n)} \in U(z; i, i^2\delta)$ for all $n > N$. This implies $j^2|x_j^{(n)} - z_j| < i^2\delta$ for $1 \leq j \leq i$ and so $|x_i^{(i)} 0 z_i| < \delta$. Thus $x_i^{(n)} \to z_i$ for all i.

Conversely, suppose $x_i^{(n)} \to z_i$ for all i. Then for all $i \in \mathbb{N}$ and $\epsilon > 0$ there is N such that $|x_i^{(n)} - z_i| < \epsilon$ for all $n > N$. In particular, for some $\delta > 0$ and $m \in \mathbb{N}$, there must be N such that $|x_i^{(n)} - z_i| < \delta/m^2$ for all $n > N$. This implies $j^2|x_j^{(n)} - z_i| < \delta$ for $1 \leq j \leq m$ and so $x_j \in U(x; m, \delta)$. Thus $x^{(n)} \in X$ converges to z in $(X, \mathcal{T})$.

$\therefore x^{(n)}$ converges to z in $(X, \mathcal{T})$ iff $x_i^{(n)} \to z_i$ for all i.

- Same setup as above. Let $V(x; \epsilon) = \{y \in X \big| |x_i - y_i| < \epsilon,\ \forall i \in \mathbb{N}\}$. If $\mathcal{M}(x) = \{V(x; \epsilon),\ \epsilon > 0\}$, is $\mathcal{M}$ a local base?

  Solution. Suppose $V \in \mathcal{M}(x)$. Then $x \in V$ since $|x_i - x_i| = 0 < \epsilon$ for all $\epsilon > 0$.

  Suppose $V_1, V_2 \in \mathcal{M}(x)$. Then $V_1 = V(x, \epsilon_1)$ and $V_2 = V(x; \epsilon_2)$ for some $\epsilon_1, \epsilon_2 > 0$. Let $V_3 = V(x; \min(\epsilon_1, \epsilon_2))$. Then for $y \in V_3$, it follows that $y \in V_1$ and $y \in V_2$ So, $V_3 \subseteq V_1 \cap V_2$.

  Suppose $V \in \mathcal{M}(x)$. Then $V = V(x; \epsilon)$ for some $\epsilon > 0$. Let $W = V(x; \epsilon/3) \subseteq V$. When $y \in W$, consider $U = V(y; \epsilon/3)$. Suppose $z \in U$, then $|z_i - x_i| \leq |z_i - y_i| + |y_i - x_i| \leq 2\epsilon/3$ for all i. This implies $U \subseteq V$.

  $\therefore \mathcal{M}(x)$ is a local base.

3. Problem 3.2.7 in Prof. Flaschka's notes, prove the following proposition:

Proposition. *Let X and Y be equipped with local bases as defined in Definition 3.2.5 and let $\mathcal{T}$, $\mathcal{S}$ denote the topologies determined on X and Y per Proposition 3.1.17. Then $f : X \to Y$ is continuous according to Definition 3.2.3 if and only if it is continuous according to Definition 3.2.5.*

Proof. First off, write out the definitions.

(3.2.3) Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$ be a function. One says that f is continuous at $x_0 \in X$ if for every open set $V \subset Y$ containing $f(x_0)$, there is an open set $U \subset X$ containing x_0, so that $x \in U$ guarantees that $f(x) \in V$.

(3.2.5) Let $\{\mathcal{N}(X) \big| x \in X\}$ and $\{\mathcal{M}(Y) \big| y \in Y\}$ be local bases on the sets X and Y. Say that a function $f : X \to Y$ is continuous at x_0 if for every $V \in \mathcal{M}(f(x_0))$ there is a $U \in \mathcal{N}(x_0)$, such that $x \in U$ implies $f(x) \in V$.

($\implies$) Assume f is continuous wrt 3.2.23, the topologies are defined as in 3.1.17:

$$U \in \mathcal{T} \text{ iff } \forall x \in U, \exists V \in \mathcal{N}(x) \text{ s.t. } V \subset U.$$

Now, if V is a neighborhood of x, $\exists W \in \mathcal{N}(x)$ s.t. $\forall x \in W$, $W \subset V$. Apply this to 3.2.23, $\forall U \in X$, $\exists$ some neighborhood V of x s.t. $V \subset U$ where $x \in V$ guarantees $f(x) \in O$ for O is open in Y. Also, every neighborhood $V \supset W$, where W is open wrt

$\mathcal{T}$ s.t. $x \in W \subset U$. Similarly, $\exists Q \in \mathcal{M}(f(x)) \in O$ s.t. Q contains an open set Ω open s.t. $f(x) \in \Omega \subset O$. Putting all of this together yields $\forall Q$, neighborhoods of $f(x_0)$, $\exists V \subset f^{-1}(Q)$ s.t. $x \in V \implies f(x) \in Q$. So 3.2.23 implies 3.2.5.

($\Longleftarrow$) Assume that 3.2.5 is true. So, for every neighborhood V of $f(x)$, $\exists$ some neighborhood $U = f^{-1}(V)$ such that for every $x \in U$, $f(x) \in V$ is guaranteed. But, by 3.1.17, inside every neighborhood V and U, there exists open sets W and O completely contained within these neighborhoods. Therefore, $x \in W \subset U \implies f(x) \in O \subset V$, and f is continuous according to 3.2.3. Therefore, 3.2.5 $\implies$ 3.2.3.

$\therefore$ 3.2.3 $\iff$ 3.2.5.

4. Problems 3.2.23 and 3.2.24 from Prof. Flaschka's notes:

   (3.2.23) Let X, Y be sets, and let f be a function with domain $\subset X$ and range $\subset Y$. Let $\mathcal{A}$ be a collection of subsets of Y. show that

$$\text{(a) } f^{-1}\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f^{-1}(A);$$

   Solution. For simplicity, we will use $\bigcup$ to imply the union over all $A \in \mathcal{A}$. We first note that if none of the sets A intersect the range of f, then both sides of this equality are trivially the empty set. Let $x \in f^{-1}(\bigcup A)$. Then $f(x) \in \bigcup A$, so $f(x) \in A$ for some $A \in \mathcal{A}$. Thus $x \in f^{-1}(A)$ for some $A \in \mathcal{A}$, so $x \in \bigcup f^{-1}(A)$ and we have $\subseteq$ containment. Now suppose $x \in \bigcup f^{-1}(A)$. Then $x \in f^{-1}(A)$ for some $A \in \mathcal{A}$, so we have $f(x) \in A$ for some $A \in \mathcal{A}$, and thus $x \in f^{-1}(\bigcup A)$. This gives us $\supseteq$ containment, and thus the sets are equal.

$$\text{(b) } f^{-1}\left(\bigcap_{A\in\mathcal{A}} A\right) = \bigcap_{A\in\mathcal{A}} f^{-1}(A);$$

   Solution. For simplicity, we will use $\bigcap$ to imply intersection over all $A \in \mathcal{A}$. First we note that if the intersection $\cap A = \emptyset$, then $f^{-1}(\emptyset) = \emptyset$, and also $\bigcap f^{-1}(A)$ must also be empty, else there exists some x s.t. $f(x) \in A$ for all A, which would imply $f(x) \in \bigcap A \neq \emptyset$. Also if $\bigcap A$ is not empty but is not in the range of f, then we also have trivially that both sets in the above equation are the empty set.

   Let $x \in f^{-1}(\bigcap A)$. Then $f(x) \in \bigcap A$, so $f(x) \in A$ for all $A \in \mathcal{A}$. Then $x \in f^{-1}(A)$ for all $A \in \mathcal{A}$, and thus $x \in \bigcap f^{-1}(A)$, so we have $\subseteq$ containment. Conversely suppose $x \in \bigcap f^{-1}(A)$. Then $f(x) \in A$ for all $A \in \mathcal{A}$, so $x \in f^{-1}(\bigcap A)$, and we have $\supseteq$ containment. Therefore the sets are equal.

$$\text{(c) } f(f^{-1}(A)) \subset A,\ A \subset Y;$$

Solution. Let $y \in f(f^{-1}(A))$. Then $y = f(x)$ for some $x \in f^{-1}(A)$, but if x is in the pullback of A, then $f(x) \in A$, so $y \in A$. Therefore we have $\subseteq$ containment.

$$\text{(d) } f^{-1}(f(B)) \supseteq B,\ B \subset X.$$

Solution. Let $x \in B$. Then $f(x) \in f(B)$, so $x \in f^{-1}(f(B))$, and we have $\supseteq$ containment.

(3.2.24)

Find examples to show that the subset symbol in (c), (d) in the last exercise cannot be replaced by a = sign.

Solution. For part c), let $A = [-1, 81]$, $f(x) = x^2$, $Y = \mathbb{R}$,

$$\begin{aligned} f^{-1}(A) &= [-9, 9] \\ f(f^{-1}(A)) &= [0, 81]. \end{aligned}$$

Since $[0, 81] \subset [-1, 81]$, $f(f^{-1}(A)) \subset A$. Therefore, $\subset$ must be used, not $\subseteq$.

For part d), let $B_1 = [-2, 2]$ and $B_2 = [0, 2]$, $f(x) = x^2$, $X = \mathbb{R}$

$$\begin{aligned} f(B_1) = f(B_2) &= [0, 4] \\ f^{-1}(f(B_1)) = f^{-1}(f(B_2)) &= [-2, 2]. \end{aligned}$$

Since $[-2, 2] \supseteq [-2, 2] \supset [0, 2]$, $f^{-1}(f(B)) \supseteq B$.

5. **First Countable local base**

$(X, \mathcal{T})$ is first countable, if it has a local base $\mathcal{N}(x)$ such that at every point $x \in X$, the collection of neighborhoods $\mathcal{N}(x)$ is countable.

(a) Show that the metric topology on a metric space (X, d) is first countable.

Solution. Let $\mathcal{T}_1$ be the metric topology on (X, d) generated by $\mathcal{N}_1(x) = \{B_\epsilon(x) | \epsilon > 0\}$. Let $\mathcal{T}_2$ be the topology generated by $\mathcal{N}_2(x) = \{B_{1/n}(x) | n \in \mathbb{N}\}$, a countable local base.

Suppose $U \in \mathcal{T}_1$. Then for all $x \in U$ there is $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U$. This implies $B_{1/n}(x) \subseteq B_\epsilon(x) \subseteq U$ for any $n \geq 1/\epsilon$ (which must be defined). So $U \in \mathcal{T}_2$

Suppose $U \in \mathcal{T}_2$. Then for all $x \in U$ there is $n \in \mathbb{N}$ s.t. $B_{1/n}(x) \subseteq U$. This implies $B_\epsilon(x) \subseteq B_{1/n}(x) \subseteq U$ for any $\epsilon \leq 1/n$. So, $U \in \mathcal{T}_1$.

It follows that $\mathbb{N}_2(x)$ generates the metric topology on (X, d) since $\mathcal{T}_1 = \mathcal{T}_2$.

$\therefore$ the metric topology on a metric space (X, d) is first countable.

(b) If $(X, \mathcal{T})$ is first countable, show that every point $x \in X$ has a countable collection of open neighborhoods $U_n \ni x$ such that $U_n \supseteq U_{n+1}$, and x is an interior point for an open set O if and only if there is an index n such that $x \in U_n \subseteq O$.

Solution. $(X, \mathcal{T})$ is first countable, so it has a countable local base $\mathcal{N}(x)$. Let $\{A_n\}$ be an enumeration of $\mathcal{N}(x)$. We will construct a sequence of nested open neighborhoods $\{U_n\}$, which give a local base equivalent to $\{A_n\}$.

From problem 7.2, for every $x \in X$, it is possible to find $(V_n \ni x) \in \mathcal{T}$ s.t. $V_n \subseteq A_n$. Let $U_1 = V_1$ and $U_{n+1} = V_{n+1} \cap U_n$. In this way, $x \in U_n \in \mathcal{T}$ and $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$.

Suppose x is an interior point for $O \in \mathcal{T}$. Then $x \in A_n \subseteq O$ for some $n \in \mathbb{N}$. But since $x \in U_n \subseteq V_n \subseteq A_n$, it follows that $x \in U_n \subseteq O$. So, there is an index n such that $x \in U_n \subseteq O$.

Conversely, suppose there is an index n such that $x \in U_n \subseteq O$ for an open set O. Then there must be $A \in \mathcal{N}(x)$ such that $A \subseteq U_n$ since $x \in U_n \in \mathcal{T}$. So, x is an interior point for an O.

$\therefore$ x is an interior point for an open set O iff there is an index n s.t. $x \in U_n \subseteq O$.

(c) With the same definitions as the previous part, show that if you construct a sequence by picking arbitrary points $y_n \in U_n$, it follows that the sequence $\{y_n\}$ converges.

Solution. I want to show that $y_n \to x$, then for any open set $O \ni x$, there exists some N_O such that for $n \geq N$, $y_n \in O$. Using our results from (b), then, for every $O \ni x$, there again exists N_O such that for $n \geq N$, $(U_n \ni x) \subset O$. Thus $y_n \in U_n \subset O$, $\forall n \geq N_O$. Therefore for any open set O which contains x, one can find points y_n that are also in O, and we say that the sequence y_n converges to x.

(d) If $(X, \mathcal{T})$ is first countable, and $(Y, \mathcal{S})$ is any topological space, show that $f : X \to Y$ is continuous if and only if it is sequentially continuous. (See Problem 3.2.16 in Prof. Flaschka's notes)

Solution. ($\implies$) Suppose f is continuous. Use the argument from c) to show that it is sequentially continuous. That is, since $y_n \to x$ (from (c)), $f(y_n) \to f(x)$, since f is continuous.

($\implies$) Show the contrapositive: if f is not continuous, then there exists some $y_n \to x$ such that $f(y_n) \not\to f(x)$.

Since f is not continuous, then there exists some open set $W \in Y$, where $f(x) \in W$ such that for every $O \ni x$, $f(O) \nsubseteq W$. Given that $(X, \mathcal{T})$ is first countable, then we have from our previous problems that there is some index such that $x \in U_n \subset O \subseteq X$, and there exists some $y_n \in U_n$. So we have that y_n converges to x, as we have shown in (c). However, since $f(U_n) \nsubseteq W$, that $f(y_n) \notin W$.
So, iff f is not continuous, there exists some $y_n \to x$, yet $f(y_n) \not\to f(x)$. $\therefore$ f is continuous iff it is sequentially continuous.

6. A set in a topological space is *closed* if it's complement is open.

   (a) If $f : X \to \mathbb{R}$ is a continuous function, show that $f^{-1}([0, 1])$ is closed in X.

Solution. From the definition of a continuous function, $\forall V \subset \mathbb{R}$, $f^{-1}(V)$ is an open subset of X. So, to show that $f^{-1}([0,1])$ is closed, consider $[0,1]^c = (-\infty, 0) \cup (1, \infty)$ which is open in $\mathbb{R}$. Since f is continuous, $\exists A$ open in X s.t. $A = f^{-1}([0,1]^c)$. Therefore, $A^c = f^{-1}([0,1])$ is closed in X.

(b) Problem 3.3.9 of Prof. Flaschka's notes.

(3.3.9)

Prove: If $(X, \mathcal{T})$ is Hausdorff, then for every $x \in X$, the set $\{x\}$ is closed. Is the converse true?

Proof. Let $x \in X$, a Hausdorff space, and let $A \subset X$. Let $x, y \in X$ such that $x \in A$ and $y \notin A$. Then for each $x \in A$ define two families of open neighborhoods of x, U_x and V_x such that $y \in V_x$. The union of all the neighborhoods U_x of x in A contain A, that is $\mathcal{U} = \bigcup_i U_{x_i} \supseteq A$.

Define $\mathcal{V} = \mathcal{U} \cap V_x$. Thus $y \in \mathcal{V}$ and $\mathcal{V} \cap A = \emptyset$, as expected. Now if we take $y \in X$ A, and let V_y be an open set of y with $V_y \cap A = \emptyset$, then we have that $X \backslash A = \bigcup_j V_{y_j}$. But, since the arbitrary union of open sets is again open, we have that $X \backslash A$ is open, thus $(X \backslash A)^c = A$ is closed, since its complement is open. Finally, setting $A = \{x\}$, we have that $\{x\}$ is closed.

$\therefore$ for every $x \in X$, the set $\{x\}$ is closed. Now, what about the converse? I claim that it is false.

Claim. *The converse of this statement is false.*

The converse is not necessarily true. Consider the topological space $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$. In this case $\{x\}$ is closed for all $x \in \mathbb{R}$ but the space is not Hausdorff.

(c) Problems 3.3.10 and 3.3.11 of Prof. Flaschka's notes.

(3.3.10)

Prove: In a Hausdorff space, every convergent sequence has a unique limit (compare Example 3.2.15).

Proof. Suppose that $(X, \mathcal{T})$ is Hausdorff, but there exists a sequence $\{x\}$ which has limits $a, b \in X$. Thus, for any open set $U \ni a$, $\exists N_U$ such that for all $n \geq N_U$, $x_n \in U$. Similarly, for an open set $V \ni b$, $\exists N_V$ such that for all $n \geq N_V$, $x_n \in V$. Then, if I choose $N = max(N_U, N_V)$, for all $n \geq N$ we have that $x_n \in U$ and $x_n \in V$, thus $x_n \in U \cap V$. But, since $(X, \mathcal{T})$ is Hausdorff, there exists open sets U, V such that $a \in U$, and $b \in V$, with $a \neq b$, and $U \cap V = \emptyset$. Clearly, x_n can not converge to b.

$\therefore$ in a Hausdorff space, every convergent sequence has a unique limit.

(3.3.11)

Let $(X, \mathcal{T})$ be first countable (Exercise 3.2.16), and suppose that every convergent sequence has a unique limit. Show that $(X, \mathcal{T})$ is Hausdorff.

Solution. Let $(X, \mathcal{T})$ be first countable and suppose that every convergent sequence has a unique limit. If $y, z \in X$ and $y \neq z$, then there are countable collections of open neighborhoods $U_n \ni y$ and $V_n \ni z$ s.t. $U_{n+1} \subseteq U_n$ and $V_{n+1} \subseteq V_n$. A sequence $\{x_n\}$ constructed by choosing $x_n \in U_n$ will converge to y regardless of the choices made, and $\{x_n\}$ must not converge to z since the limits are unique. This implies that there is N s.t. $x_n \notin V_n$, and so $U_n \cap V_n = \emptyset$ since the points x_n are arbitrary, for $n > N$. This yields $y \in U_n$, $z \in V_n$, and $U_n \cap V_n = \emptyset$ for any $n > N$.

$\therefore$ the topological space $(X, \mathcal{T})$ is Hausdorff.